Physics 498/MMA — **Mathematical Methods in Physics I** — Prof. M. Stone
Handout 3 — http://w3.physics.uiuc.edu/∼m-stone5 — 2117 ESB
Fall 2007 — HOMEWORK 3 — University of Illinois

1) Test functions and distributions: Read the sections on *distributions* in chapter two of the lecture notes, then do the following problems:

a) Let $f(x)$ be a smooth function.

i) Show that $f(x)\delta(x) = f(0)\delta(x)$. Deduce that

$$\frac{d}{dx}[f(x)\delta(x)] = f(0)\delta'(x).$$

ii) We might also have used the product rule to conclude that

$$\frac{d}{dx}[f(x)\delta(x)] = f'(x)\delta(x) + f(x)\delta'(x).$$

By integrating both against a test function, show this expression for the derivative of $f(x)\delta(x)$ is equivalent to that in part i).

b) Let $\varphi(x)$ be a test function. Using the definition of the *principal part integrals*, show that

$$\frac{\partial}{\partial t}\left\{P\int_{-\infty}^{\infty}\frac{\varphi(x)}{(x-t)}\,dx\right\} = P\int_{-\infty}^{\infty}\frac{\varphi(x)-\varphi(t)}{(x-t)^2}\,dx$$

in two different ways:

i) Fix the value of the cutoff ϵ. Differentiate the resulting ϵ-regulated integral, taking care to include the terms arising from the t dependence of the limits at $x = t \pm \epsilon$.

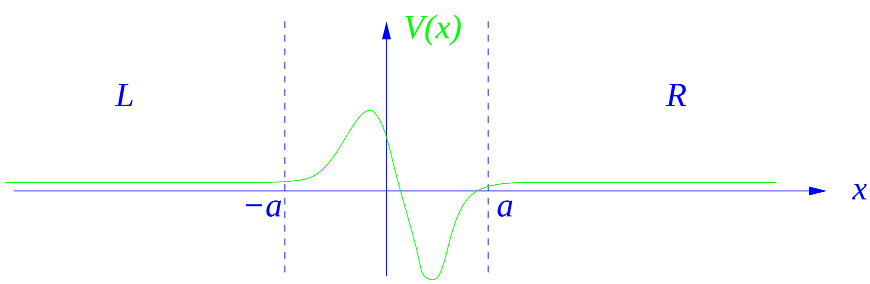
ii) First make a change of variables $y = x - t$, so that the singularity is fixed at $y = 0$. Now differentiate with respect to t. Next integrate by parts to take the derivative off φ and onto the singular factor. (Take care to include the boundary contributions.) Finally change back to the original x, t variables.

Both methods should give the same result!

2) One-dimensional scattering theory: Consider the one-dimensional Schrödinger equation

$$-\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

where $V(x)$ is zero except in a finite interval $[-a, a]$ near the origin.



Let L denote the left asymptotic region, $-\infty < x < -a$, and similarly let R denote $\infty > x > a$. For $E = k^2$ and $k > 0$ there will be scattering solutions of the form

$$\psi_k(x) = \begin{cases} e^{ikx} + r_L(k)e^{-ikx}, & x \in L, \\ t_L(k)e^{ikx}, & x \in R, \end{cases}$$

describing waves incident on the potential $V(x)$ from the left. For $k < 0$ there will be solutions with waves incident from the right

$$\psi_k(x) = \begin{cases} t_R(k)e^{ikx}, & x \in L, \\ e^{ikx} + r_R(k)e^{-ikx}, & x \in R. \end{cases}$$

The wavefunctions in $[-a, a]$ will naturally be more complicated. Observe that $[\psi_k(x)]^*$ is also a solution of the Schrödinger equation.

By using properties of the Wronskian, show that:

a) $|r_{L,R}|^2 + |t_{L,R}|^2 = 1$,
b) $t_L(k){=}t_R(-k)$.
c) Deduce from parts a) and b) that $|r_L(k)| = |r_R(-k)|$.
d) Take the specific example of $V(x) = \lambda\delta(x - b)$ with $|b| < a$. Compute the transmission and reflection coefficients and hence show that $r_L(k)$ and $r_R(-k)$ may differ by a phase.

3) Reduction of Order: Sometimes additional information about the solutions of a differential equation enables us to reduce the order of the equation, and so solve it.

a) Suppose that we know that $y_1 = u(x)$ is one solution to the equation

$$y'' + V(x)y = 0.$$

By trying $y = u(x)v(x)$ show that

$$y_2 = u(x) \int^x \frac{d\xi}{u^2(\xi)}$$

is also a solution of the differential equation. Is this new solution ever merely a constant mutiple of the old solution, or must it be linearly independent? (Hint: evaluate the Wronskian $W(y_2, y_1)$.)

b) Suppose that we are told that the product, y_1y_2, of the two solutions to the equation $y'' + p_1y' + p_2y = 0$ is a constant. Show that this requires $2p_1p_2 + p_2' = 0$.
c) By using ideas from part b) or otherwise, find the general solution of the equation

$$(x + 1)x^2y'' + xy' - (x + 1)^3y = 0.$$

4) Normal forms and the Schwarzian derivative: We saw in class that if y obeys a second-order linear differential equation

$$y'' + p_1y' + p_2y = 0$$

then we can make always make a substitution $y = w\tilde{y}$ so that $\tilde{y}$ obeys an equation without a first derivative:

$$\tilde{y}'' + q(x)\tilde{y} = 0.$$

Suppose $\psi(x)$ obeys a Schrödinger equation

$$\left(-\frac{1}{2}\frac{d^2}{dx^2} + [V(x) - E]\right)\psi = 0.$$

a) Make a smooth and invertable change of independent variable by setting $x = x(z)$ and find the second order differential equation in z obeyed by $\psi(z) \equiv \psi(x(z))$. Find the $\tilde{\psi}(z)$ that obeys an equation with no first derivative. Show that this equation is

$$\left(-\frac{1}{2}\frac{d^2}{dz^2} + (x')^2[V(x(z)) - E] - \frac{1}{4}\{x, z\}\right)\tilde{\psi}(z) = 0,$$

where the primes denote differentiation with respect to z, and

$$\{x, z\} \equiv \frac{x'''}{x'} - \frac{3}{2}\left(\frac{x''}{x'}\right)^2$$

is called the *Schwarzian* derivative of x with respect to z. Schwarzian derivatives play an important role in conformal field theory and string theory.

b) Now combine a sequence of maps $x \to z \to w$ to establish *Cayley's identity*

$$\left(\frac{dz}{dw}\right)^2 \{x, z\} + \{z, w\} = \{x, w\}.$$

(Hint: If this takes you more than one line, you are missing the point.)