1) Missing State: In Homework Set 4 you found that the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} - 2\,\text{sech}^2 x\right)\psi = E\,\psi$$

has eigensolutions

$$\psi_k(x) = e^{ikx}(-ik + \tanh x)$$

with eigenvalue $E = k^2$.

- For x large and positive $\psi_k(x) \approx A\,e^{ikx}e^{i\delta(k)}$, while for x large and negative $\psi_k(x) \approx A\,e^{ikx}e^{-i\delta(k)}$, the (complex) constant A being the same in both cases. Express $\delta(k)$ as the inverse tangent of an algebraic expression in k.
- Impose periodic boundary conditions $\psi(-L/2) = \psi(+L/2)$ where $L \gg 1$. Find the allowed values of k and hence an explicit expression for the k-space density, $\rho(k) = \frac{dn}{dk}$, of the eigenstates.
- Compare your formula for $\rho(k)$ with the corresponding expression, $\rho_0(k) = L/2\pi$, for the eigenstate density of the zero-potential equation and compute the integral

$$\Delta N = \int_{-\infty}^{\infty}\{\rho(k) - \rho_0(k)\}dk.$$

- Deduce that one eigenfunction has gone missing from the continuum and presumably become a localized bound state. (You will have found an explicit expression for this localized eigenstate in Homework Set 4.)

2) Continuum Completeness: Consider the differential operator

$$\hat{L} = -\frac{d^2}{dx^2}, \qquad 0 \le x < \infty$$

with self-adjoint boundary conditions $\psi(0)/\psi'(0) = \tan\theta$ for some fixed angle θ.

- Show that when $\tan\theta < 0$ there is a single normalizable negative-eigenvalue eigenfunction localized near the origin, but none when $\tan\theta > 0$.
- Show that there is a continuum of positive-eigenvalue eigenfunctions of the form $\psi_k(x) = \sin(kx + \delta(k))$ where the phase shift δ is found from

$$e^{i\delta(k)} = \frac{1 + ik\tan\theta}{\sqrt{1 + k^2\tan^2\theta}}.$$

- Write down (no justification required) the appropriate completeness relation

$$\delta(x - x') = \int \frac{dn}{dk} N_k \psi_k(x)\psi_k(x')\,dk + \sum_{\text{bound}} \psi_n(x)\psi_n(x')$$

with an explicit expression for the product (not the separate factors) of the density of states and the normalization constant N_k, and with the correct limits on the integral over k.

- Confirm that the ψ_k continuum on its own, or together with the bound state when it exists, form a complete set. You will do this by evaluating the integral

$$I(x, x') = \frac{2}{\pi} \int_0^\infty \sin(kx + \delta(k)) \sin(kx' + \delta(k))\, dk$$

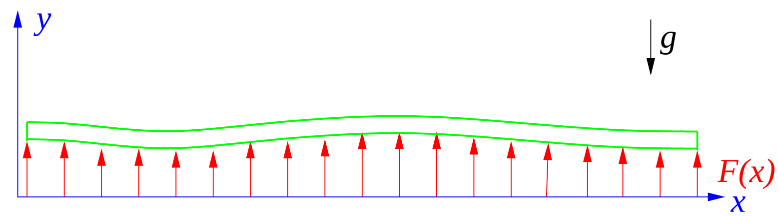
and interpreting the result. You will need the following standard integral

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \frac{1}{1 + k^2 t^2} = \frac{1}{2|t|} e^{-|x|/|t|}.$$

To get full credit, you must show how the bound state contribution switches on and off with θ. The modulus signs are essential for this.

3) Fredholm Alternative:

A heavy elastic bar with uniform mass m per unit length lies almost horizontally. It is supported by a distribution of upward forces $F(x)$.



The shape of the bar, $y(x)$, can be found by minimizing the energy

$$U[y] = \int_0^L \left\{ \frac{1}{2} \kappa (y'')^2 - (F(x) - mg) y \right\} dx,$$

which gives (homework 2!) the equation

$$\hat{L} y \equiv \kappa \frac{d^4 y}{dx^4} = F(x) - mg, \quad y'' = y''' = 0 \quad \text{at} \quad x = 0, L.$$

- Show that the boundary conditions are such that the operator $\hat{L}$ is self-adjoint with respect to an inner product with weight function 1.
- Find the zero modes which span the null space of $\hat{L}$.
- If there are n linearly independent zero modes, then the codimension of the range of $\hat{L}$ is also n. Using your explicit solutions from the previous part, find the conditions that must be obeyed by $F(x)$ for a solution of $\hat{L} y = F - mg$ to exist. What is the physical meaning of these conditions?
- The solution to the equation and boundary conditions is not unique. Is this non-uniqueness physically reasonable? Explain.