

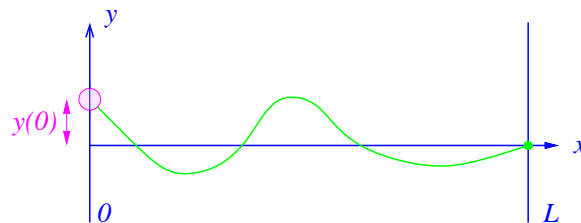
Do question one, and then as many other questions as you can. Try to answer entire questions. Little, if any, credit will be given for fragmentary answers.

1) Green Function: Consider the boundary value problem

$$-y'' = f(x), \quad y'(0) = y(1) = 0.$$

- Construct the explicit Green function appropriate to this problem. (5 points)
- Use your Green function to write down the solution of the boundary value problem as the sum of two explicit integrals over complementary components of the unit interval. (5 points)
- Evaluate the x derivative of your solution, $y(x)$, and confirm that y obeys both boundary conditions. (5 points)
- Take one further derivative of your $y(x)$ and confirm that it does indeed solve the original problem. (5 points)

2) Bead and string: A bead of mass M is free to slide up and down the y axis.



A bead connected to a string.

It is attached to the $x = 0$ end of a string in such a way that the Lagrangian for the string-bead system is

$$L = \frac{1}{2}M[\dot{y}(0)]^2 + \int_0^L \left\{ \frac{1}{2}\rho\dot{y}^2 - \frac{1}{2}Ty'^2 \right\} dx.$$

Here ρ is mass per unit length of the string and T its tension.

- Write down the Euler-Lagrange equations which come from varying $y(x)$ subject to the condition that $y(L)$ is fixed. You should end up with an equation for the string in the interval $0 < x \leq L$, and another governing the motion of the bead. (10 points)
- Write down the *First Integral* appropriate to your system of Euler-Lagrange equations. Verify, from your equations of motion for the bead and string, that your expression for the first integral is independent of time. (10 points)

3) Lagrange Multipliers: Use the method of undetermined multipliers to find the stationary point(s) of the function $F(x, y, z) = x^2 + 3xy + y^2 + z^2$ subject to the constraint that $x + y + z = 1$. (10 points)

4) Fourier sine transform:

- a) Write down the *completeness relation* (**not** the orthogonality condition) obeyed by the functions $\varphi_k(x) = \sin kx$ on the interval $0 \leq x < \infty$. (5 points)
- b) If the sine transformation of $f(x)$ is defined by

$$F(k) = \int_0^{\infty} f(x) \sin kx \, dx,$$

show how to recover $f(x)$ from $F(k)$ by means of an inverse transformation. (5 points)

5) Adjoint operator: A linear operator is defined by

$$L = \frac{d^4}{dx^4}$$

with domain of definition

$$\mathcal{D}(L) = \{y, y^{(4)} \in L^2[0, 1] : y(0) = y'''(0) = y(1) = y'''(1) = 0\}.$$

What is

- a) the differential operator (5 points),
- b) the domain of definition (boundary conditions), (5 points)

of the adjoint operator L^\dagger ?

In the course of determining the domain of L^\dagger you may make use of the identity

$$\frac{d}{dx} (y_1''' y_2 - y_1'' y_2' + y_1' y_2'' - y_1 y_2''') = y_1'''' y_2 - y_1 y_2''''$$

without first bothering to prove it.

— End —