Solutions to Homework Set 0

Differential calculus: The point of the exercise was to make sure that you know how to differentiate integrals with respect to their limits:

$$\frac{d}{da}\int_a^b f(x)\,dx = -f(a), \qquad \frac{d}{db}\int_a^b f(x)\,dx = f(b),$$

and in general

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)\,dx = f(b(t))\frac{\partial b}{\partial t} - f(a(t))\frac{\partial a}{\partial t} + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)\,dx.$$

Once you have this under control, the first problem is plug and chug:
From

$$y(x) = \frac{\sin\omega(x-L)}{\omega\sin\omega L}\int_0^x f(t)\sin\omega t\,dt + \frac{\sin\omega x}{\omega\sin\omega L}\int_x^L f(t)\sin\omega(t-L)\,dt$$

we get

$$y'(x) = \frac{\cos\omega(x-L)}{\sin\omega L}\int_0^x f(t)\sin\omega t\,dt + \frac{\cos\omega x}{\sin\omega L}\int_x^L f(t)\sin\omega(t-L)\,dt.$$

The two terms arising from the derivative of the integration limits have cancelled against each other. When we differentiate again, the two pieces arising from differentiating the factors outside the integral assemble to give $-\omega^2 y(x)$. The two terms from differentiating the integrals are

$$y''(x) + \omega^2 y(x) = f(x)\frac{\cos\omega(x-L)\sin\omega x - \cos\omega x\sin\omega(x-L)}{\sin\omega L}.$$

The addition formula for $\sin(A+B)$, now shows that this is equal to $f(x)$ as required.

For the second problem we have

$$\begin{aligned}
F'(x) &= K(0)f(x) + \int_0^x \partial_x K(x-y)f(y)\,dy \\
&= K(0)f(x) - \int_0^x f(y)\partial_y K(x-y)\,dy \\
&= K(0)f(x) - \int_0^x \partial_y[f(y)K(x-y)]\,dy + \int_0^x f'(y)K(x-y)\,dy \\
&= K(0)f(x) - K(0)f(x) + f(0)K(x) + \int_0^x f'(y)K(x-y)\,dy \\
&= f(0)K(x) + \int_0^x K(x-y)f'(y)\,dy.
\end{aligned}$$

He was not quite right therefore—unless $f(0)$ happens to be zero.

Integral Calculus: I hope that you did not proceed as follows:

$$I(\lambda,\mu)=\int_0^\infty t^{-1}(e^{-\lambda t}-e^{-\mu t})\,dt \stackrel{?}{=} \int_0^\infty t^{-1}e^{-\lambda t}\,dt-\int_0^\infty t^{-1}e^{-\mu t}\,dt.$$

Then, manipulating blindly (*i.e.* shutting your eyes to the fact that the integrals on the right-hand side are divergent), a change of variable makes everything appear independent of λ, and μ. From this we conclude that $I(\lambda,\mu)$ is zero. This is nonsense.
Two legitimate approaches to the problem come to mind:

i) Differentiate under the integral sign:

$$\begin{aligned}\frac{d}{d\mu}I(\lambda,\mu) &= \int_0^\infty \frac{d}{d\mu}\left(\frac{e^{-\lambda t}-e^{-\mu t}}{t}\right)dt\\ &= \int_0^\infty e^{-\mu t}\,dt\\ &= 1/\mu.\end{aligned}$$

Integrating up, we find that

$$I(\lambda,\mu)=\ln\mu+c.$$

The constant c is determined by noting that $I=0$ when $\lambda=\mu$, whence

$$I(\lambda,\mu)=\ln\mu-\ln\lambda.$$

ii) Put limits on the divergent integrals:

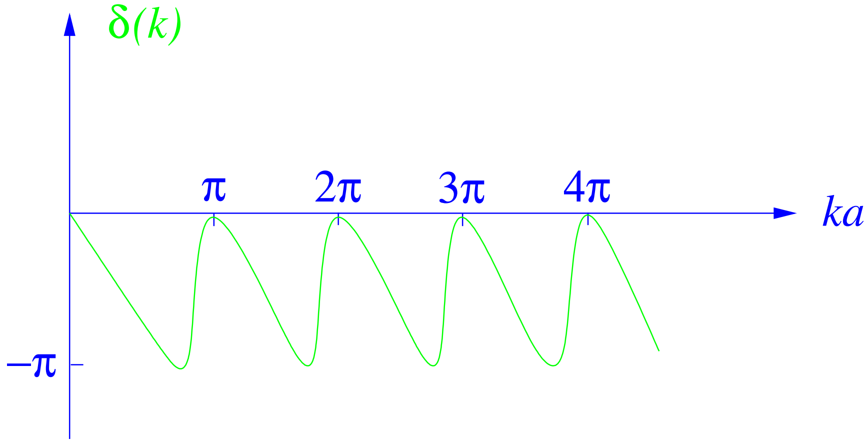
$$\begin{aligned}\int_0^\infty \frac{e^{-\lambda t}-e^{-\mu t}}{t}\,dt &= \lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty \frac{e^{-\lambda t}-e^{-\mu t}}{t}\,dt\right\}\\ &= \lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty \frac{e^{-\lambda t}}{t}\,dt-\int_\epsilon^\infty \frac{e^{-\mu t}}{t}\,dt\right\}\\ &= \lim_{\epsilon\to 0}\left\{\int_{\lambda\epsilon}^\infty \frac{e^{-t}}{t}\,dt-\int_{\mu\epsilon}^\infty \frac{e^{-t}}{t}\,dt\right\}\\ &= \lim_{\epsilon\to 0}\left\{\int_{\lambda\epsilon}^{\mu\epsilon} \frac{e^{-t}}{t}\,dt\right\}\\ &= \lim_{\epsilon\to 0}\left\{\int_{\lambda\epsilon}^{\mu\epsilon} \frac{1}{t}\,dt\right\}\\ &= \lim_{\epsilon\to 0}\{\ln(\epsilon\mu)-\ln(\epsilon\lambda)\}\\ &= \ln\mu-\ln\lambda.\end{aligned}$$

When f is differentiable either method can be used to evaluate *Frullani's integral* and find that

$$F(\lambda,\mu)=\int_0^\infty \frac{f(\lambda t)-f(\mu t)}{t}\,dt=f(0)\ln(\mu/\lambda).$$

If $f(t)$ is continuous but not differentiable then we are restricted to the the second method.

Trigonometry: You need realize that λ is completelely negligeable when $\cot ka$ is close to to $\pm\infty$. With no λ, and choosing the branches of the arc-cotangent so as to preserve continuity, we have $\cot^{-1}(\cot ka) = ka$ for all values of ka. The only effect of a large (positive) λ is to make the arc-cotangent hang around $\cot^{-1}(+\infty) = n\pi$ for a while, until $\cot ka$ becomes sufficiently negative to drag the $+\infty$ down to finite values. A plot of $\cot^{-1}(\lambda + \cot ka)$ is therefore a staircase with flat treads at $\delta = n\pi$ and sharp risers just before $ka = n\pi$, the risers being sharper and closer to $ka = n\pi$ the larger is λ.



Sketch plot of $\delta(k) = -ka + \cot^{-1}(\lambda + \cot ka)$.

Mathematica is unable to deal with the infinities, and gives plots containing unphysical discontinuities.

Partial derivatives: Using the chain rule we have

$$\frac{\partial}{\partial t} = \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} + \frac{\partial z}{\partial t}\frac{\partial}{\partial z} = \frac{\partial}{\partial \tau} - U\frac{\partial}{\partial z},$$

and

$$\frac{\partial}{\partial x} = \frac{\partial z}{\partial x}\frac{\partial}{\partial z} + \frac{\partial \tau}{\partial x}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial z}.$$

We therefore find that

$$i\hbar\left(\frac{\partial}{\partial \tau} - U\frac{\partial}{\partial z}\right)\tilde{\psi} = -\frac{\hbar^2}{2m}\frac{\partial^2\tilde{\psi}}{\partial z^2} + V(z)\tilde{\psi}.$$

The potential is now time independent. This modified Schrödinger equation is solved by plugging in $\tilde{\psi} = e^{i\alpha z + \beta\tau}\psi(z,\tau)$ and finding α and β by requiring the coefficients of $\tilde{\psi}$ and $\tilde{\psi}'$ to be zero. Resubstituting, $\tau = t$, $z = x - Ut$ leads to

$$\tilde{\psi}(x,t) = e^{imUx/\hbar - i\frac{1}{2}mU^2t/\hbar}\psi(x - Ut, t).$$

The wave-function does not transform as a scalar function, but is instead an observer dependent object.

Matrix Algebra:

i) Suppose that T has an eigenvector $\mathbf{x}$ with eigenvalue μ, so $T\mathbf{x} = \mu\mathbf{x}$. Then

$$0 = (T - \lambda I)^N \mathbf{x} = (\mu - \lambda)^N \mathbf{x}.$$

Since $\mathbf{x}$ is non-zero we see that $(\lambda - \mu)^N = 0$, but if the N-th power of a number is zero, the number itself must be zero. Thus λ is the only possible eigenvalue. If the matrix representing T were diagonalizable, all the numbers on the diagonal would have to be λ and the diagonalized matrix would be $\mathbf{T} = \lambda\mathbf{I}$. This matrix, and hence the linear operator T that it represents, would then obey $(\mathbf{T} - \lambda\mathbf{I})^1 = 0$ — but, unless $N = 1$, this is in contradiction to what we were told about T. Thus T *cannot* be diagonalized.

ii) That there exists a vector $\mathbf{e}_1$ such that $(T - \lambda I)^N \mathbf{e}_1 = 0$, but no lesser power of $(T - \lambda I)$ kills $\mathbf{e}_1$, is simply a restatement of what we are given: all vectors are killed by $(T - \lambda I)^N$, but if there were no vector that survived $(T - \lambda I)^{N-1}$, then we would have $(T - \lambda I)^{N-1} = 0$. We are told, however, that $(T - \lambda I)^{N-1}$ is not zero.

iii) To prove linear independence we must start from the definition of linear independence. Suppose, therefore, that we can find N numbers λ_i such that

$$\lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \ldots + \lambda_N \mathbf{e}_N = 0.$$

Act on this expression by $(T - \lambda I)^{N-1}$,to get

$$0 = \lambda_1 (T - \lambda I)^{N-1} \mathbf{e}_1,$$

all other terms being killed. Since we know that $(T - \lambda I)^{N-1} \mathbf{e}_1 \neq 0$, we must have $\lambda_1 = 0$. Now act with $(T - \lambda I)^{N-2}$, to get

$$0 = \lambda_2 (T - \lambda I)^{N-2} \mathbf{e}_2 = \lambda_2 (T - \lambda I)^{N-1} \mathbf{e}_1,$$

all other terms being killed, or being zero because $\lambda_1 = 0$. We now deduce that $\lambda_2 = 0$. Proceeding in this manner we deduce that all the λ_i are zero. Thus the $\mathbf{e}_i$ are indeed linearly independent.

iv) In the $\mathbf{e}_i$ basis the matrix becomes

$$T \to \mathbf{T} = \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & \\ & & & & \ddots \end{pmatrix},$$

i.e. a matrix with λ's on the diagonal, and 1's immediately above, all other entries being zero. This is called the *Jordan canonical form* of the non-diagonalizable operator.